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Thermally driven motion of strongly heated fluids

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Abstract—A formal, unified method is used to derive two sets of approximate equations governing the thermally driven motion of strongly heated fluids. Account is taken of the temperature dependence of the fluid properties. The limits considered correspond to: (a) a strongly heated, shallow fluid; and (b) a strongly heated, deep liquid with a small coefficient of volume expansion. Two additional limits that are considered correspond to: (c) a weakly heated, deep fluid; and (d) a weakly heated, shallow fluid. For cases (a), (c) and (d), use is made of two basic expansion parameters: the nondimensionalized depth ε of the fluid and the nondimensionalized strength δ' of the heat source distribution. Case (a) corresponds to $\varepsilon \ll 1$; the resulting equations govern the temperature, density and pressure to lowest order. The lowest order pressure is uniform but time dependent. Case (c) corresponds to $\delta' \ll 1$. The unperturbed state for this case is taken to be steady and quiescent; this restricts it to the so-called adiabatic state. Case (d) is obtained from the results of case (c) by also taking $\varepsilon \ll 1$. The resulting equations constitute the Boussinesq approximation, and are valid independent of the relative magnitude of δ' and ε. An arbitrary temperature gradient of order δ' can be added to the unperturbed state of a weakly heated fluid. For the case of a weakly heated, shallow ideal gas maintained at constant volume and heated throughout the volume, the equations account for the effect of perturbation pressure on perturbation density and on compression heating. Case (b) yields an energy equation that is significantly different from that for case (d). When applied to a liquid under normal conditions, case (b) leads to the Boussinesq equations with temperature-dependent fluid properties. The sets of equations obtained all are based on a characteristic velocity that is vanishingly small compared with the speed of sound. As a result, the equations do not contain acoustic waves, and their numerical solution is no more difficult than that of incompressible flow equations.

1. INTRODUCTION

THE EQUATIONS governing the thermally driven motion of fluids have been the subject of many investigations. In principle, the motion can be determined by solving the fundamental equations of fluid dynamics. However, the solution of these equations presents formidable problems in practice. Various authors have proposed simplified, approximate sets of equations that are applicable to many problems of practical interest. There exist sets applicable to gases and liquids, weakly heated either by a prescribed temperature difference or by a distributed source, with viscous effects either restricted to thin layers or dominating the entire flow. Most of these are restricted to 'shallow' fluids, i.e. fluids with a depth much smaller than the characteristic hydrostatic length scale. Finally, there are sets applicable to shallow, strongly heated fluids, i.e. fluids in which the heating causes large perturbations in density.

Classical papers presenting simplified, approximate sets of equations are those of Oberbeck [1, 2] and of Boussinesq [3]. The results obtained by the latter have become widely accepted, and are referred to as the Boussinesq approximation. The earlier result of Oberbeck [1] is nearly the same as the Boussinesq approximation, the only difference being that Oberbeck retained density variations in the continuity equation. These variations now are known to be of higher order, and hence negligible. An early application of the Boussinesq approximation was that of Rayleigh [4], who considered the growth of modes of disturbance in a thin layer of liquid heated from below. Jeffreys [5] showed that for the case of a gas, the thermal driving term in the Boussinesq approximation is proportional to the excess of the actual temperature gradient over the adiabatic temperature gradient. Ogura and Phillips [6] analyzed the time scales involved in the Boussinesq approximation. They concluded that these pertain to gravity waves, while excluding high-frequency acoustic waves.

During recent years, various authors have re-examined the precise conditions under which the Boussinesq approximation is valid. Spiegel and Veronis [7] re-examined the case of a gas by introducing conditions involving orders of magnitude. They concluded that the validity conditions are: (a) the vertical dimension of the gas is much less than any scale height; and (b) the motion-induced fluctuations in density and pressure do not exceed, in order of magnitude, the total static variations of these quantities. Gebhart [8] and Plate [9] used similar methods and arrived at similar conclusions. Milhaljan [10] used a perturbation approach to re-examine the case of a thin layer of liquid. He showed rigorously that the Boussinesq approximate equations can be obtained as the lowest order of an expansion in powers of two nondimensional parameters. One of these parameters

NOMENCLATURE

B, C, C_1, C_2	constants order of 1
c_m	$(p_m/\rho_m)^{1/2}$
c_v	specific heat at constant volume, nondimensionalized by c_{vm}
c_p	specific heat at constant pressure, nondimensionalized by c_{pm}
g	acceleration of gravity
Gr	Grashof number, see equation (11a) and (11b)
$\bar{k}$	unit vector in z-direction
K	isothermal compressibility, $\rho^{-1}(\partial\rho/\partial p)_\theta$
L	length scale
p_m	pressure
P	pressure, nondimensionalized by p_m
Pr	Prandtl number, $\mu c_p/\kappa$
Q	rate of heat addition per unit volume nondimensionalized by Q_m
t	time, nondimensionalized by L/β
T	temperature, nondimensionalized by θ_m
$\bar{V}$	velocity, nondimensionalized by β
z	vertical coordinate, nondimensionalized by L.

Greek symbols

α	coefficient of volume expansion, $-\rho^{-1}(\partial\rho/\partial\theta)_p$
β	characteristic velocity
γ	ratio of specific heats
Γ	volume, nondimensionalized by L^3
δ	$\alpha_m\theta_m$ times nondimensionalized strength δ' of heat source [equation (6)] or of temperature difference [equation (9)]; $\beta^2/(gL)$
δ'	$\delta/(a_m\theta_m)$
ε	nondimensionalized depth, gL/c_m^2
θ	temperature
$\Delta\theta$	temperature difference
κ	thermal conductivity, nondimensionalized by κ_m
μ	dynamic viscosity, nondimensionalized by μ_m
ν	kinematic viscosity, μ/ρ, nondimensionalized by ν_m
ρ	density, nondimensionalized by ρ_m
τ	viscous part of stress tensor, nondimensionalized by $\mu_m\beta/L$
ϕ	$c_m^2/(c_{pm}\theta_m)$.

Subscripts

m	constant dimensional quantities, used in the nondimensionalization of equations
z	component in the z-direction
0, 1, 2	order in perturbation expansion.

is the nondimensionalized coefficient of thermal expansion, while the other involves the thermal diffusivity and the specific heat; neither parameter explicitly contains the scale height. A similar approach was followed by Malkus [11, 12], who emphasized the case of a perfect gas. Malkus used two other small, nondimensional parameters: the ratio of the physical depth to the hydrostatic adiabatic depth, and the maximum relative temperature variation. Malkus took the hydrostatic adiabatic field as the unperturbed state. Calder [13] showed that the temperature gradient of the unperturbed state may be chosen arbitrarily, and specified the resulting changes in the Boussinesq equations. Cordon and Velarde [14] presented a detailed analysis of the approaches followed by Milhaljan [10] and Malkus [12]. Following Malkus, they took the hydrostatic adiabatic field as the unperturbed state. Cordon and Velarde found a solution for this field that is applicable to a certain class of liquids and gases. They choose two expansion parameters that were of the same order numerically for typical situations of interest. The work of Gray and Giorgini [15] also is applicable to both liquids and gases. They nondimensionalized the basic equations, and considered the values of the dimensionless parameters that occur in various terms. Based on this consideration, they presented equations for two orders of approximation. One of these applies to shallow fluids, and leads to the set of equations originally presented by Boussinesq [3]. The other one is valid also for deep fluids, and contains additional terms. Gray and Giorgini referred to these sets as the 'strict' and the 'extended' Boussinesq approximation, respectively. They presented plots showing the regions of validity of each set for both water and air. Dutton and Fichtl [16] and Gough [17] derived results for weakly heated, deep fluids. These results were extended by Kovshov [18] to the case that the reference field has time-dependent variations in the direction of gravity. Forester and Emery [19] considered free convective flows of fluids with large density gradients in the absence of acoustic waves. Rehm and Baum [20] derived an approximate set of equations for the thermally driven flows of gases with large density gradients in the absence of heat conduction and viscous effects. Paolucci [21] showed that approximate equations governing thermal convection in gases can be derived by considering the limit of small Mach number. The relations between the results of previous workers and those of the present paper are discussed in some detail in the Appendix.

Equations applicable to the thermally driven motion of geophysical fluids in the earth's mantle have been

presented by Turcotte *et al.* [22–24], Velarde and Cordon [25], and Oxburgh and Turcotte [26]. Geophysical fluids have a very large viscosity and are at a very high pressure. The motion of such fluids can best be treated by scaling the dimensional quantities so as to account for the dominance of viscous effects over inertial effects. Fluids of this kind are left out of the account in the present work.

The main purpose of the present paper is to consider the motion of strongly rather than weakly heated gases and liquids. In other words, the temperature and density variations are allowed to be large. Approximate equations applicable to this case are derived in the limit of a small ratio of the physical depth to the hydrostatic scale depth (strongly heated, shallow fluid, Section 3). The resulting equations are no more difficult to solve numerically than the Boussinesq equations.

The method used also is applied to fluids of arbitrary depth. These can again be treated by requiring only one parameter to be small. This parameter is either the maximum relative temperature variation (Section 4, applicable to both liquids and gases), or the coefficient of thermal expansion nondimensionalized by the maximum temperature variation (Section 6, applicable to strongly heated, deep liquids). The resulting equations are another generalization of the Boussinesq approximation. For a liquid as well as for a shallow gas maintained at constant average pressure, the equations of Section 4 reduce to the Boussinesq approximation (Section 5).

2. BASIC EQUATIONS

The problem under consideration is governed by the fundamental equations of fluid dynamics, expressing conservation of mass, momentum and energy. Appropriate forms of these equations are given in many places, e.g. in Bird *et al.* [27, Table 10.4-1]. The equations can be written in the following nondimensionalized form:

$$\partial\rho/\partial t + \nabla\cdot(\rho\bar{V}) = 0 \quad \text{(continuity)} \tag{1}$$

$$\rho\left[\frac{\partial\bar{V}}{\partial t} + (\bar{V}\cdot\nabla)\bar{V}\right] = -\frac{1}{\delta\varepsilon}\nabla P - \bar{k}\frac{\rho}{\delta} - \frac{\nu_m}{\beta L}[\nabla\cdot\tau] \quad \text{(momentum)} \tag{2}$$

$$\rho c_p\left[\frac{\partial T}{\partial t} + (\bar{V}\cdot\nabla)T\right] = \frac{1}{Pr_m}\frac{\nu_m}{\beta L}\nabla\cdot(\kappa\nabla T) + \alpha\theta\phi\left[\frac{\partial P}{\partial t} + (\bar{V}\cdot\nabla)P\right] + \frac{Q_m L}{\rho_m c_{pm}\theta_m\beta}Q - \delta\varepsilon\phi\frac{\nu_m}{\beta L}(\tau:\nabla\bar{V}) \quad \text{(energy)}. \tag{3}$$

The various symbols are defined in the nomenclature. It has been assumed that there is a constant gravitational acceleration g in the negative z-direction. The components of the stress tensor τ are given by Bird *et al.* [27, Table 3.4], for several coordinate systems. It should be noted that the characteristic velocity β is as yet unspecified, while $\delta = \beta^2/(gL)$, $\varepsilon = gL/c_m^2$ and $\phi = c_m^2/(c_{pm}\theta_m)$. The parameter ε represents the ratio of the physical depth L to the characteristic hydrostatic depth c_m^2/g. Under 'normal' conditions ($p \approx 0.1$ MPa, $\theta \approx 300$ K, $g \approx 10\,\text{m s}^{-2}$), the latter depth is of the order of 10 km for gases like air, and of the order of 10 m for liquids like water. The parameter $\phi = (\gamma_m - 1)/\gamma_m$ for an ideal gas; for a liquid under normal conditions it is of the order of 10^{-4}. It is assumed throughout this paper that ϕ is at most of the order of 1. The product $\alpha\theta$ equals unity for an ideal gas, and is of the order of 0.03–0.3 for most liquids under normal conditions. In the expansion procedures of the following Sections 3–5, $\alpha_m\theta_m$ is considered to be a number of order 1; in Section 6, it is assumed to be of the same order as the small parameter $\alpha_m\Delta\theta$.

Equations (1)–(3) must be supplemented by the thermal and caloric equations of state. These are supposed to be given in the form

$$\rho = \rho(P, T) \tag{4}$$

$$c_p = c_p(T). \tag{5}$$

Furthermore, the transport coefficients are supposed to be given in the form $\mu = \mu(T)$, $\kappa = \kappa(T)$. The neglect of the dependence of c_p, μ and κ on P is justified for most cases of practical interest; if necessary, this dependence can readily be taken into account also.

The motion is supposed to be driven by either a heat source distribution $Q_m Q$, or by a prescribed temperature difference $\Delta\theta$ at the boundaries. In the former case, a consistent set of equations is obtained by setting the parameter δ equal to $\alpha_m\theta_m$ times the nondimensionalized strength of the heat source:

$$\delta = \alpha_m Q_m L/(\rho_m c_{pm}\beta). \tag{6}$$

It then follows that

$$\delta = \left(\frac{\alpha_m Q_m}{\rho_m c_{pm}}\right)^{2/3}\left(\frac{L}{g}\right)^{1/3} \tag{7}$$

$$\beta = \left(\frac{\alpha_m g L^2 Q_m}{\rho_m c_{pm}}\right)^{1/3}. \tag{8}$$

If $Q_m = 0$, but $\Delta\theta \neq 0$, equation (6) is replaced by

$$\delta = \alpha_m\Delta\theta \tag{9}$$

which leads to

$$\beta = (\alpha_m\Delta\theta g L)^{1/2}. \tag{10}$$

In either case, it follows that $\nu_m/(\beta L) = Gr_m^{-1/2}$, where the Grashof number is defined as

$$Gr_m = \frac{L^2}{\nu_m^2}\left(\frac{\alpha_m g L^2 Q_m}{\rho_m c_{pm}}\right)^{2/3} \tag{11a}$$

or

$$Gr_m = \alpha_m g L^3\Delta\theta/\nu_m^2 \tag{11b}$$

as appropriate. While it is assumed that either $Q_m = 0$ or $\Delta\theta = 0$, the method used can readily be extended to cases where both Q_m and $\Delta\theta$ are non-zero. It should be

Table 1. Values of the parameters ε and δ'

	ε	$\delta' = \delta/\alpha_m\theta_m$
Strongly heated, shallow fluid (Section 3)	$\ll 1$	$O(1)$
Weakly heated, deep fluid (Section 4)	$O(1)$	$\ll 1$
Weakly heated, shallow fluid (Section 5)	$\ll 1$	$\ll 1$
Strongly heated, deep liquid with small coefficient of thermal expansion (Section 6)	$O(1)$	$\ll (\alpha_m\theta_m)^{-1}$

noted that the ratio of the characteristic velocity β to the speed of sound a is given by $\beta/a = (\delta\varepsilon)^{1/2}(c_m/a)$, where the factor c_m/a is of order 1 for gases, and much smaller than 1 for liquids. As a consequence, satisfaction of the inequality $(\delta\varepsilon)^{1/2} \ll 1$ guarantees that $\beta \ll a$.

The magnitude of the parameter $Gr_m^{-1/2}$ appearing in equation (2) is an indication of the thickness of the viscous boundary layers that may be present in the flow. Similarly, the magnitude of the parameter $Pr_m^{-1}\, Gr_m^{-1/2}$ appearing in equation (3) is an indication of the thickness of the thermal boundary layers. It is assumed throughout this paper that $Gr_m^{-1/2}$ is at most of order 1, so that the flow is not dominated by viscous effects. With this assumption, the scaling used results in equations with variables that are all of order 1.

The various cases considered in the following sections are summarized in Table 1. Table 2 lists the values of ϕ, $\alpha\theta$ and Kp for an ideal gas, as well as typical values for a liquid.

3. STRONGLY HEATED, SHALLOW FLUID

A strongly heated, shallow fluid here is defined as one subject to the conditions $\varepsilon \ll 1$, $\delta = C_1\alpha_m\theta_m$, where C_1 is a constant of order 1. The corresponding temperature variations are of order 1. The condition on δ implies that either $[Q_m/(\rho_m c_{pm}\theta_m)]^{2/3}(L/g)^{1/3}$ or $\Delta\theta/\theta_m$ is of order 1. The corresponding characteristic velocity is $\beta = (C_1\alpha_m\theta_m gL)^{1/2}$. Expanding in powers of ε yields

$$P = P_0(\bar{r},t)+\varepsilon P_1(\bar{r},t)+\ldots, \quad \rho = \rho_0(\bar{r},t)+O(\varepsilon),$$
$$T = T_0(\bar{r},t)+O(\varepsilon), \quad \bar{V} = \bar{V}_0+O(\varepsilon),$$
$$c_p = c_{p0}(T_0)+O(\varepsilon), \quad \kappa = \kappa_0(T_0)+O(\varepsilon),$$
$$\mu = \mu_0(T_0)+O(\varepsilon), \quad \gamma = \gamma_0(T_0)+O(\varepsilon),$$
$$\tau = \tau_0(\mu_0,\bar{V}_0)+O(\varepsilon), \quad \alpha\theta = (\alpha\theta)_0+O(\varepsilon).$$

Here, use was made of the Taylor series expansion

$$f[T(\varepsilon)] = f(T_0+\varepsilon T_1+\ldots)$$
$$= f(T_0)+\varepsilon T_1\, df(T_0)/dT_0+\ldots$$
$$= f_0(T_0)+\varepsilon f_1(T_0,T_1)+\ldots$$

from which it follows that $f_0(T_0) = f(T_0)$, $f_1(T_0,T_1) = T_1\, df(T_0)/dT_0$. Equation (2) yields to order ε^{-1}

$$P_0 = P_0(t). \tag{12}$$

The nondimensionalized heat source distribution Q is of order 1 by definition. Also by definition $c_{p0}(1) = \kappa_0(1) = \mu_0(1) = 1$. Equations (1)–(3) become to order ε^0

$$\frac{\partial\rho_0}{\partial t}+\nabla\cdot(\rho_0\bar{V}_0) = 0 \tag{13}$$

$$\rho_0\left[\frac{\partial\bar{V}_0}{\partial t}+(\bar{V}_0\cdot\nabla)\bar{V}_0\right] = -\frac{1}{C_1\alpha_m\theta_m}\nabla P_1 - \bar{k}\frac{\rho_0}{C_1\alpha_m\theta_m}-\frac{1}{Gr_m^{1/2}}[\nabla\cdot\tau_0] \tag{14}$$

$$\rho_0 c_{p0}\left[\frac{\partial T_0}{\partial t}+(\bar{V}_0\cdot\nabla)T_0\right] = \frac{1}{Pr_m\, Gr_m^{1/2}}\nabla\cdot(\kappa_0\nabla T_0) + (\alpha\theta)_0\phi\frac{dP_0}{dt}+C_1Q. \tag{15a}$$

Comparison of equation (15a) with equation (3) shows that the viscous dissipation term has disappeared, and that an important simplification has been achieved in the pressure term. The latter term no longer is coupled to the pressure occurring in the momentum equation. Without this simplification, computational schemes based on the vorticity–stream function method (see, e.g. Roache [28]), and on marching forward in time require that the pressure field $P(\bar{r},t)$ be extracted from the momentum equation at each time step. Computation of the pressure term in equation (15a) is straightforward for an ideal gas (see below). For a liquid, the term can ordinarily be neglected. Computation of the term with ∇P_1 in equation (14) can be avoided by taking the curl of this equation [28].

For an *ideal gas*, the equation of state in the present notation is $P = \rho T$. This yields to lowest order $P_0 = \rho_0 T_0$. Using this result together with equation (13) to replace dP_0/dt in equation (15a) yields, after

Table 2. Values of the parameters ϕ, $\alpha\theta$ and Kp

	ϕ	$\alpha\theta$	Kp
Ideal gas	$(\gamma_m-1)/\gamma_m$	1	1
Liquid (typical)	$\approx 10^{-3}\cdot(p_m$ in MPa)	$\approx 10^{-1}$	$\approx 10^{-3}\cdot(p_m$ in MPa)

some algebra

$$\rho_0\left[\frac{\partial T_0}{\partial t}+(\bar{V}_0\cdot\nabla)T_0\right]=(\gamma_0-1)\times\left[\frac{\gamma_m}{\gamma_m-1}\frac{1}{Pr_m\,Gr_m^{1/2}}\nabla\cdot(\kappa_0\nabla T_0)-P_0(\nabla\cdot\bar{V}_0)+\frac{\gamma_m}{\gamma_m-1}C_1Q\right]. \quad (15b)$$

This form of the equation allows computation of $\partial T_0/\partial t$ for given $T_0(\bar{r},t)$, $\bar{V}_0(\bar{r},t)$, $\rho_0(\bar{r},t)$ and $Q(\bar{r},t)$. If the gas is contained in a fixed volume Γ, conservation of total mass is expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \rho_0\,\mathrm{d}\Gamma=0.$$

Application of this relation together with the ideal gas law leads to

$$\frac{\mathrm{d}P_0}{\mathrm{d}t}=P_0\frac{\int_\Gamma T_0^{-2}(\partial T_0/\partial t)\,\mathrm{d}\Gamma}{\int_\Gamma T_0^{-1}\,\mathrm{d}\Gamma} \quad (16)$$

which allows computation of $\mathrm{d}P_0/\mathrm{d}t$. With both $\partial T_0/\partial t$ and $\mathrm{d}P_0/\mathrm{d}t$ known, $\partial\rho_0/\partial t$ can be found from the relation

$$\frac{\partial\rho_0}{\partial t}=\frac{1}{T_0}\frac{\mathrm{d}P_0}{\mathrm{d}t}-\frac{P_0}{T_0^2}\frac{\partial T_0}{\partial t}. \quad (17)$$

The set of equations (13)–(17) represents a compressible analog of the Boussinesq approximate equations. Important features of this analog are the appearance of zero-order quantities, only (except for the term with the perturbation pressure P_1 in the momentum equation), and the appearance of a pressure-dependent term in the energy equation. Neither equations (13)–(17) nor the Boussinesq equations have the ability to describe sound waves, or phenomena related to such waves. In the case of equations (13)–(17), this is a consequence of the independence of the pressure P_0 on $\bar{r}$. As a result of the absence of sound waves, numerical solution schemes that are explicit in time are not subject to the Courant–Friedrichs–Lewy condition on the time step (cf. Roache [28]).

For a *liquid* under 'normal' conditions ($p_m \simeq 0.1$ MPa, $\theta_m \simeq 300$ K), the term with ϕ in equation (15a) can be neglected. Furthermore, the equation of state can be written as

$$\rho_0=\exp\left(-\int_1^{T_0}\alpha\theta_m\,\mathrm{d}T_0'\right) \quad (18a)$$

where a term $\int_1^{P_0}Kp_m\,\mathrm{d}P_0'$ in the argument of the exponential has been neglected. This is justified because Kp is of the order of 10^{-4} for liquids under normal conditions. The dependence of α on T follows from the equation of state, equation (4). For the special case that α is independent of T, equation (18a) simply becomes

$$\rho_0=\exp\left[-\alpha_m\theta_m(T_0-1)\right]. \quad (18b)$$

As mentioned before, the numerical solution of the equations obtained by the vorticity–stream function method involves taking the curl of equation (14). This means taking derivatives of ρ_0 in the term with $\bar{k}$. The resulting terms are of order 1 even if $\alpha_m\theta_m$ is small.

4. WEAKLY HEATED, DEEP FLUID

If $\delta' \equiv \delta/(\alpha_m\theta_m) \ll 1$, use can be made of the following expansions

$$\begin{aligned}P&=P_0(\varepsilon;z)+\delta'P_1(\varepsilon;\bar{r},t)+O(\delta'^2)\\ \rho&=\rho_0(\varepsilon;z)+\delta'\rho_1(\varepsilon;\bar{r},t)+O(\delta'^2)\\ T&=T_0(\varepsilon;z)+\delta'T_1(\varepsilon;\bar{r},t)+O(\delta'^2).\end{aligned} \quad (19)$$

The leading terms of these expansions represent the unperturbed state, which depends on both ε and the vertical coordinate z, but is taken to be independent of time t. The velocity in the unperturbed state is taken to be zero. The higher order terms P_1, ρ_1, T_1, etc. are taken to be dependent on position $\bar{r}$ and time t, as well as on ε. For a shallow fluid ($\varepsilon \ll 1$), each term can be expanded in powers of ε; however, this expansion is postponed until results for a deep fluid have been obtained. For the latter fluid ε is of order 1, with the understanding that the characteristic pressure p_m and the characteristic density ρ_m are to be evaluated at the bottom of the fluid in its hydrostatic adiabatic state. Additional expansions are

$$\begin{aligned}\bar{V}&=\bar{V}_1+O(\delta'),\quad \kappa=\kappa_0(T_0)+O(\delta')\\ \tau&=\tau_0(\mu_0,\bar{V}_1)+O(\delta'),\quad \mu=\mu(T_0)+O(\delta')\\ c_p&=c_{p0}(T_0)+O(\delta'),\quad \alpha\theta=(\alpha\theta)_0(T_0)+O(\delta').\end{aligned} \quad (20)$$

The lowest order term in the expansion of $\bar{V}$ is denoted by $\bar{V}_1$ because it does not belong to the unperturbed hydrostatic state described by P_0, ρ_0 and T_0. In the expressions just given, the possible dependence of the zero-order quantities on P_0 is neglected.

Equating terms of order $1/\delta'$ in equation (2) yields

$$\mathrm{d}P_0/\mathrm{d}z=-\varepsilon\rho_0. \quad (21)$$

Using this result together with equation (3) to order $(\delta')^0$ yields

$$\rho_0c_{p0}V_{1z}\frac{\mathrm{d}T_0}{\mathrm{d}z}=\frac{1}{Pr_m\,Gr_m^{1/2}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa_0\frac{\mathrm{d}T_0}{\mathrm{d}z}\right)-(\alpha\theta)_0\varepsilon\phi\rho_0V_{1z}. \quad (22)$$

The unperturbed temperature T_0 must be independent of the thermally induced velocity V_{1z}. This requires that the first term on the RHS of equation (22) is zero

$$(\mathrm{d}/\mathrm{d}z)(\kappa_0\,\mathrm{d}T_0/\mathrm{d}z)=0 \quad (23a)$$

and that the remaining terms are equal. The latter condition is automatically satisfied if $V_{1z}=0$. This corresponds to the case in which small perturbations do not induce a velocity field, i.e. to an unperturbed situation that is stably stratified. Excluding this case

from further consideration, the condition becomes

$$c_{p0}\,\mathrm{d}T_0/\mathrm{d}z = -(\alpha\theta)_0\varepsilon\phi. \tag{23b}$$

This equation can be restated in the form $\mathrm{d}s_0/\mathrm{d}z = 0$, since the entropy s_0 of the unperturbed state follows from $\mathrm{d}s_0 = c_{p0}\,\mathrm{d}T_0 + (\alpha\theta)_0\rho_0^{-1}\,\mathrm{d}P_0$, while $\mathrm{d}P_0$ is given by equation (21). Consistency of the perturbation scheme adopted requires that equations (23a) and (23b) be satisfied simultaneously. This is possible if, and only if, the ratio $c_{p0}/(\alpha_0\theta_0\kappa_0)$ is a constant. The solution then is given by

$$\int_1^{T_0(z)} \frac{c_{p0}}{(\alpha\theta)_0}\,\mathrm{d}T_0' = -\varepsilon\phi z. \tag{24}$$

The dependence of ρ_0 and P_0 on z must follow from integrating equation (21) while taking account of the equation of state, equation (4). The resulting solution is the hydrostatic adiabatic state. The temperature and density fields of this state correspond to those of a fluid particle rising or descending isentropically in the corresponding pressure field. It should be noted that the adiabatic state must be maintained by a constant heat flux in the vertical direction; in this sense, it actually is non-adiabatic. States with a temperature gradient larger than that given by equation (23b) are stably stratified. On the other hand, states with smaller temperature gradients are unstable. Temperature or density perturbations of the latter states are subject to amplification, and result in self-excited motion with amplitudes that are not necessarily small. The present theory is not applicable to such motion, but is limited to motions forced by the perturbation, with amplitudes of the same order as the perturbation.

For the case $\alpha_m\theta_m\varepsilon\phi \ll 1$ [shallow gas, or a liquid of depth $L \ll c_{pm}/(g\alpha_m)$], the hydrostatic adiabatic state can be written as

$$\begin{aligned} T_0 &= 1 - \alpha_m\theta_m\varepsilon\phi z + O(\alpha_m\theta_m\varepsilon\phi)^2 \\ \rho_0 &= 1 - \varepsilon\phi z[(K_m p_m/\phi) - (\alpha_m\theta_m)^2] + O(\alpha_m\theta_m\varepsilon\phi)^2 \\ P_0 &= 1 - \varepsilon\left\{z - \varepsilon\phi\frac{z^2}{2}[(K_m p_m/\phi) - (\alpha_m\theta_m)^2]\right\} \\ &\quad + O(\alpha_m\theta_m\varepsilon\phi)^2. \end{aligned} \tag{25}$$

For an ideal gas, $K_m p_m = \alpha_m\theta_m = 1$, so that the terms in the square brackets become $1/(\gamma_m - 1)$. For a liquid, Kp/ϕ typically is of order 1, while $\alpha\theta$ typically is of order 0.03–0.3, as mentioned before. For example, for water at 0.1 MPa and 300 K, $Kp/\phi = 0.56$, $\alpha\theta = 0.08$. Thus, the first term in the square brackets typically is dominant over the second one. The square bracket term may also be written as $K_m p_m c_{vm}/(c_{pm}\phi)$.

Returning to the general case $\alpha_m\theta_m\varepsilon\phi = O(1)$, it is found that the perturbed state is governed by equations (1)–(3) to order $(\delta')^0$, $(\delta')^0$ and $(\delta')^1$, respectively

$$\nabla\cdot(\rho_0\bar{V}_1) = 0 \tag{26}$$

$$\rho_0\left[\frac{\partial\bar{V}_1}{\partial t} + (\bar{V}_1\cdot\nabla)\bar{V}_1\right] = -\frac{1}{\alpha_m\theta_m\varepsilon}\nabla P_1 - \bar{k}\frac{\rho_1}{\alpha_m\theta_m} - \frac{1}{Gr_m^{1/2}}[\nabla\cdot\tau_0] \tag{27}$$

$$\begin{aligned} &\rho_0 c_{p0}\left[\frac{\partial T_1}{\partial t} + \bar{V}_1\cdot\nabla T_1\right] - \varepsilon\phi V_{1z} \\ &\quad\times\left[(\alpha\theta)_0\rho_1 - (\alpha\theta)_1\rho_0 + (\alpha\theta)_0\rho_0\frac{c_{p1}}{c_{p0}}\right] \\ &= \frac{1}{Pr_m\,Gr_m^{1/2}}\left[\nabla\cdot(\kappa_0\nabla T_1) + \frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa_1\frac{\mathrm{d}}{\mathrm{d}z}T_0\right)\right] \\ &\quad + (\alpha\theta)_0\phi\left[\frac{\partial P_1}{\partial t} + \bar{V}_1\cdot\nabla P_1\right] + Q - \alpha_m\theta_m\varepsilon\phi \\ &\quad\times\frac{1}{Gr_m^{1/2}}(\tau_0 : \nabla\bar{V}_1). \end{aligned} \tag{28}$$

Here, $\kappa_1 = T_1\,\mathrm{d}\kappa(T_0)/\mathrm{d}T_0$, $c_{p1} = T_1\,\mathrm{d}c_p(T_0)/\mathrm{d}T_0$ and $(\alpha\theta)_1 = T_1\,\mathrm{d}[\alpha\theta(T_0)]/\mathrm{d}T_0$. In the derivation of equation (28), two terms with $\bar{V}_2\cdot\nabla$ are found to cancel in view of equations (21) and (22). The parameter $\alpha_m\theta_m\varepsilon\phi$ occurring as a coefficient in the dissipation term of equation (28) has previously been discussed by Ostrach [29]. Ostrach pointed out that this parameter can be large for fluids in high speed rotating machinery (large g), and for fluids near the critical point (large α). Turcotte *et al.* [23] called it the dissipation parameter, and investigated its influence on the thermally driven motion of a fluid with zero compressibility.

Before obtaining further results, it is helpful to specify whether the fluid is a gas or a liquid.

4.1. *Ideal gas*

For an ideal gas with constant specific heat and thermal conductivity ($c_{p0} = \kappa_0 = 1$), the unperturbed state can be written in a simple closed form

$$T_0 = 1 - \frac{\gamma_m - 1}{\gamma_m}\varepsilon z, \quad P_0 = T_0^{\gamma_m/(\gamma_m - 1)}, \quad \rho_0 = T_0^{1/(\gamma_m - 1)}. \tag{29}$$

When use is made of these relations, it is consistent to set $\mu_0 = 1$ in the evaluation of τ_0.

The perturbation density ρ_1 in equations (27) and (28) follows from equating terms of order δ' in the expansion of the ideal gas equation $P = \rho T$

$$\rho_1 = \rho_0\left(\frac{P_1}{P_0} - \frac{T_1}{T_0}\right). \tag{30}$$

As mentioned before, $\phi = (\gamma_m - 1)/\gamma_m$ for an ideal gas. Furthermore $\alpha\theta = (\alpha\theta)_0 = 1$, $(\alpha\theta)_1 = 0$. If the gas is maintained at constant volume to order δ', conservation of mass requires that $\int_\Gamma \rho_1\,\mathrm{d}\Gamma = 0$. This condition determines the reference level of the perturbation pressure P_1 (i.e. the arbitrary constant that results when P_1 is evaluated from ∇P_1 in the vorticity–stream function method).

The set of equations (26)–(30) constitutes another compressible analog of the Boussinesq approximation, applicable to a weakly heated, ideal gas with depth of order 1. Important features of this analog are the appearance in the energy equation of two terms with P_1 and a term with $\rho_1 V_{1z}$, as well as the dissipation term

4.2. *Liquid*

Substituting expansions (19) into the equation of state, equation (4), yields to order δ'

$$\rho_1 = \rho_0(K_0 p_m P_1 - \alpha_0\theta_m T_1). \tag{31}$$

For a liquid with $L \ll c_{pm}/(g\alpha_m)$, the terms with ϕ and Kp_m in equations (25)–(28) and (31) can be neglected. The resulting equations are

$$T_0 = \rho_0 = 1, \quad P_0 = 1 - \varepsilon z \tag{32}$$

$$\nabla \bar{V}_1 = 0 \tag{33}$$

$$\frac{\partial \bar{V}_1}{\partial t} + (\bar{V}_1 \cdot \nabla)\bar{V}_1 = -\frac{1}{\alpha_m\theta_m\varepsilon}\nabla P_1 + \bar{k}T_1 - \frac{1}{Gr_m^{1/2}}[\nabla \cdot \tau_0] \tag{34}$$

$$\frac{\partial T_1}{\partial t} + (\bar{V}_1 \cdot \nabla)T_1 = \frac{1}{Pr_m\, Gr_m^{1/2}}\nabla^2 T_1 + Q. \tag{35}$$

The quantities c_{p0} and κ_0 both are equal to one in view of the first of equations (32). Equations (33)–(35) constitute the well-known Boussinesq approximation, derived here for a weakly heated ($\delta' \ll 1$), deep (ε of order 1) liquid, subject to the restriction $L \ll c_{pm}/(g\alpha_m)$ (i.e. $\alpha_m\theta_m\phi \ll 1$).

5. WEAKLY HEATED, SHALLOW FLUID

If $\delta' \ll 1$ *and* $\varepsilon \ll 1$, use can be made of the following expansions for the first-order quantities appearing in equation (19)

$$\begin{aligned} P_1(\varepsilon; \bar{r}, t) &= P_{10}(\bar{r}, t) + \varepsilon P_{11}(\bar{r}, t) + O(\varepsilon^2) \\ \rho_1(\varepsilon; \bar{r}, t) &= \rho_{10}(\bar{r}, t) + O(\varepsilon) \\ T_1(\varepsilon; \bar{r}, t) &= T_{10}(\bar{r}, t) + O(\varepsilon). \end{aligned} \tag{36}$$

Similarly, the lowest order quantities in equation (20) can be expanded as $\bar{V}_1 = \bar{V}_{10} + O(\varepsilon)$, etc. It is assumed that ϕ is at most of order 1, so that equations (25) yield $T_{00} - \rho_{00} = P_{00} - 1$. Equation (27) now yields to order ε^{-1}

$$\nabla P_{10} = 0 \tag{37}$$

with the solution $P_{10} = P_{10}(t)$. The form of $P_{10}(t)$ depends on the conditions at which the gas is maintained (see below). Equations (26)–(28) yield to order ε^0, ε^0 and ε, respectively

$$\nabla \cdot \bar{V}_{10} = 0 \tag{38}$$

$$\frac{\partial \bar{V}_{10}}{\partial t} + (\bar{V}_{10} \cdot \nabla)\bar{V}_{10} = -\frac{1}{\alpha_m\theta_m}\nabla P_{11} - \bar{k}\frac{\rho_{10}}{\alpha_m\theta_m} - \frac{1}{Gr_m^{1/2}}[\nabla \cdot \tau_{00}] \tag{39}$$

$$\frac{\partial T_{10}}{\partial t} + (\bar{V}_{10} \cdot \nabla)T_{10} = \frac{1}{Pr_m\, Gr_m^{1/2}}\nabla^2 T_{10} + (\alpha\theta)_{00}\phi\frac{dP_{10}}{dt} + Q. \tag{40}$$

If the fluid is an *ideal gas*, the perturbation density ρ_{10} in equation (39) follows from equation (30) to order ε^0

$$\rho_{10} = P_{10} - T_{10}. \tag{41a}$$

If the gas is maintained at a constant volume Γ, conservation of mass to order δ' is expressed by the condition

$$\int_\Gamma \rho_{10}\, d\Gamma = 0.$$

Substitution of equation (41a) into this condition yields

$$P_{10} = \Gamma^{-1}\int_\Gamma T_{10}\, d\Gamma.$$

Combining this relation with equation (41a) provides the result

$$\rho_{10} = -T_{10} + \Gamma^{-1}\int_\Gamma T_{10}\, d\Gamma. \tag{41b}$$

If, on the other hand, the gas is maintained at constant average pressure to order δ', so that $P_{10} = 0$, equation (41a) becomes

$$\rho_{10} = -T_{10}. \tag{41c}$$

Equation (41b) reduces to equation (41c) in the limit that T_{10} is negligibly small everywhere except in a thin layer, such as the free convection layer on a vertical surface.

Equations (38)–(41a) with P_{10} set equal to zero again constitute the Boussinesq approximation. The need for letting P_{10} be a function of time if the gas is maintained at constant volume rather than at constant pressure is not mentioned in any of the papers referenced. The resulting extra term in equation (41b) as compared with equation (41c) represents the effect of a uniform rise in perturbation pressure on perturbation density. The term with dP_{10}/dt in equation (40) represents the effect of uniform compression heating.

If the fluid is a *liquid* under normal conditions, the term with ϕ in equation (40) can be neglected. Furthermore, equation (31) yields to order ε^0: $\rho_{10} = -\alpha_m\theta_m T_{10}$. This again leads to the Boussinesq approximation.

The validity of equations (38)–(40) is independent of the relative magnitude of δ' and ε. For $\delta' \ll \varepsilon(\ll 1)$, the thermally driven perturbations are much smaller than the corresponding static variations. For $\varepsilon \ll \delta'$ ($\ll 1$), the static variations may be neglected, and the unperturbed state simply becomes the uniform state $P_0 = \rho_0 = T_0 = 1$.

If $\delta' = C\varepsilon(\ll 1)$, where C is a constant of order 1, an *alternative treatment* consists of setting

$$\begin{aligned} P &= P_0(z) + \varepsilon P_1(\bar{r}, t) + \varepsilon^2 P_2(\bar{r}, t) + O(\varepsilon^3) \\ \rho &= \rho_0(z) + \varepsilon\rho_1(\bar{r}, t) + O(\varepsilon^2) \\ T &= T_0(z) + \varepsilon T_1(\bar{r}, t) + O(\varepsilon^2). \end{aligned} \tag{42}$$

Equation (2) then yields to order ε^{-2}: $\nabla P_0 = 0$, with the

solution $P_0 = 1$. Equation (3) becomes to order ε^0

$$\rho_0 c_{p0} V_{1z} \frac{dT_0}{dz} = \frac{1}{Pr_m\, Gr_m^{1/2}} \frac{d}{dz}\left(\kappa_0 \frac{dT_0}{dz}\right).$$

Because T_0 must be independent of V_{1z}, the solution of this equation is $T_0 = \text{const.} = 1$. It then follows from the equation of state that $\rho_0 = 1$, so that the unperturbed state is uniform to lowest order in ε. Next, equation (2) yields to order ε^{-1}: $\partial P_1/\partial z = -1$, with the solution

$$P_1 = -z + f(t). \tag{43}$$

Equations (1)–(3) become to order ε^0, ε^0 and ε, respectively

$$\nabla \cdot \bar{V}_1 = 0 \tag{44}$$

$$\frac{\partial \bar{V}_1}{\partial t} + (\bar{V}_1 \cdot \nabla)\bar{V}_1 = -\frac{1}{\alpha_m \theta_m C}\nabla P_2 - \bar{k}\frac{\rho_1}{\alpha_m \theta_m C} - \frac{1}{Gr_m^{1/2}}[\nabla \cdot \tau_0] \tag{45}$$

$$\frac{\partial T_1}{\partial t} + (\bar{V}_1 \cdot \nabla)T_1 + \alpha_m \theta_m \phi V_{1z} = \frac{1}{Pr_m\, Gr_m^{1/2}}\nabla^2 T_1 + \alpha_m \theta_m \phi \frac{df}{dt} + CQ. \tag{46}$$

For a *liquid* under normal conditions the terms with ϕ can again be neglected, while $\rho_1 = -\alpha_m \theta_m T_1$. This leads once again to the Boussinesq approximation.

For an *ideal gas* maintained at a steady pressure to order ε, $f(t) = 0$ so that $P_1 = -z$. The equation of state then yields to order ε

$$\rho_1 = -z - T_1. \tag{47a}$$

If the gas is maintained at constant volume to order ε, conservation of mass to order ε requires $\int_\Gamma \rho_1\, d\Gamma = \text{const.}$ This constant can be determined from the initial conditions, using the relation $\rho_1 = P_1 - T_1$. It follows that

$$\rho_1 = -z - T_1 + \Gamma^{-1}\int_\Gamma (z + T_1)\, d\Gamma + \Gamma^{-1}\int_\Gamma \rho_1\, d\Gamma \tag{47b}$$

$$\frac{df}{dt} = \Gamma^{-1}\int_\Gamma \frac{\partial T_1}{\partial t}\, d\Gamma. \tag{48}$$

In the relations just given, the first order quantities P_1, ρ_1 and T_1 represent both the unperturbed state and the thermally driven perturbations. An arbitrary temperature gradient of order ε can be added to the unperturbed state by setting $T_1 = T_1' + Bz$, where B is a constant of order 1. This results in replacing T_1 by T' and ϕV_{1z} by $(\phi + B)V_{1z}$ in equation (46). For the adiabatic state, $B = -\phi = -(\gamma_m - 1)/\gamma_m$ [cf. the first of equations (25)], and the term with V_{1z} disappears. The possibility of taking a state with an arbitrary temperature gradient of order ε as the unperturbed state was utilized by Rayleigh [4], and further discussed by Calder [13].

6. STRONGLY HEATED, DEEP LIQUID WITH SMALL COEFFICIENT OF THERMAL EXPANSION

In the previous sections, $\alpha_m \theta_m$ was considered to be a number of order 1. An alternative approach is to consider this quantity to be of the same order as the parameter δ, defined by either equation (6) or equation (9), and taking δ instead of δ' as the small parameter

$$\delta = C_1 \alpha_m \theta_m \ll 1. \tag{49}$$

Here, C_1 again is a constant of order 1, defined by either

$$Q_m L/(\rho_m c_{pm} \theta_m \beta) = C_1 \tag{50a}$$

or

$$\Delta\theta/\theta_m = C_1. \tag{50b}$$

The value of C_1 is limited by the condition $C_1 \ll (\alpha_m \theta_m)^{-1}$. Equations (49) and (50) are equivalent to equations (6) and (9), and the characteristic velocity again follows from $\beta = (\delta g L)^{1/2}$. Denoting ε by C_2 to indicate that this parameter also is of order 1, use may be made of the following expansions

$$\begin{aligned} P &= P_0(C_2; z) + \delta P_1(C_2; \bar{r}, t) + O(\delta^2) \\ \rho &= \rho_0(C_2; z) + \delta \rho_1(C_2; \bar{r}, t) + O(\delta^2) \\ T &= T_0(C_2; \bar{r}, t) + O(\delta) \\ \bar{V} &= \bar{V}_1(C_2; \bar{r}, t) + O(\delta), \quad \text{etc.} \end{aligned} \tag{51}$$

Equation (2) then becomes to order δ^{-1}

$$dP_0/dz = -C_2 \rho_0, \tag{52}$$

while the differential equation of state $d \ln \rho = K p_m\, dP - \alpha \theta_m\, dT$ yields to order δ^0

$$d \ln \rho_0 = K_0 p_m\, dP_0. \tag{53}$$

The term $K_0 p_m$ is retained in order to extend the validity of the results to very deep liquids. If the isothermal compressibility $K = \text{const.} = K_m$, equations (52) and (53) have the solution

$$\begin{aligned} P_0 &= 1 - (K_m p_m)^{-1} \ln (1 + C_2 K_m p_m z) \\ \rho_0 &= (1 + C_2 K_m p_m z)^{-1}. \end{aligned} \tag{54}$$

Assuming that ϕ is at most of order 1, the perturbed state in the general case with temperature-dependent fluid properties is given by equations (1)–(3) to order δ^0

$$\nabla \cdot (\rho_0 \bar{V}_1) = 0 \tag{55}$$

$$\rho_0\left[\frac{\partial \bar{V}_1}{\partial t} + (\bar{V}_1 \cdot \nabla)\bar{V}_1\right] = -\frac{1}{C_2}\nabla P_1 - \bar{k}\rho_1 - \frac{1}{Gr_m^{1/2}}[\nabla \cdot \tau_0] \tag{56}$$

$$\rho_0 c_{p0}\left[\frac{\partial T_0}{\partial t} + \bar{V}_1 \cdot \nabla T_0\right] = \frac{1}{Pr_m\, Gr_m^{1/2}}\nabla \cdot (\kappa_0 \nabla T_0) + C_1 Q. \tag{57}$$

Equating terms of order δ in the differential equation of state and integrating yields (again taking $K = K_m$)

$$\rho_1 = \rho_0 \left[K_m p_m P_1 - C_1^{-1} \int_1^{T_0} (\alpha_0/\alpha_m)\, dT_0' \right]. \quad (58)$$

The constant of integration was determined by requiring $\rho_1 = 0$ for $P_1 = 0, T_1 = 1$. Equations (55) and (56) are identical in form with equations (26) and (27), respectively. On the other hand, equation (57) is significantly different from equation (28). The principal differences are the presence of a term with ϕV_{1z}, of two terms with P_1, and of a viscous dissipation term in the latter but not in the former. The assumption $c_{p0} = 1$ made in writing down equation (28) is not appropriate for equation (57), which specifically applies to cases with large temperature variations.

If the liquid is at normal conditions ($K_m p_m \ll 1$) even though it is still deep ($\varepsilon = C_2$), equations (54)–(58) are further simplified. The resulting equations still incorporate temperature-dependent properties μ_0, c_{p0}, κ_0 and α_0, but otherwise constitute the Boussinesq approximation.

7. BRIEF SUMMARY

Various sets of approximate equations have been obtained, using either the nondimensionalized depth ε or the nondimensionalized strength δ' of the heat source as small parameters. The first set obtains in the limit $\varepsilon \to 0$, and describes the motion of a strongly heated, shallow fluid [Section 3, equations (13)–(17)]. Compared with the fundamental equations of fluid dynamics, the main simplification achieved is that the pressure appearing in the energy equation is uniform, and is uncoupled from the pressure occurring in the momentum equation. The latter pressure occurs in the form of a gradient, and can be eliminated by taking the curl of the momentum equation. A second set of equations is applicable to a liquid, in the limit that $\alpha_m \theta_m \delta' \to 0$ (Section 6). This allows inclusion of the case of a strongly heated liquid having a small coefficient of volume expansion [$\delta' = O(1), \alpha_m \theta_m \ll 1$]. The resulting energy equation [equation (57)] is found to be significantly different from the corresponding equation of a third set, applicable in the limit $\delta' \to 0, \alpha_m \theta_m = O(1)$. The latter set describes the motion of a weakly heated, deep fluid (Section 4). The unperturbed state is taken to be time-independent and quiescent. It is found that consistency of the perturbation scheme requires the unperturbed state to be the adiabatic state [equations (21)–(24)]. The Boussinesq approximation is recovered by carrying out a second expansion (Section 5). For a gas maintained at constant volume, and heated throughout the volume, the perturbation pressure P_{10} affects the perturbation density ρ_{10} [equation (41a)]. For this case, compression heating due to the rise in perturbation pressure also must be taken into account [term with dP_{10}/dt in equation (40)]. If the heating parameter is of the same order as the depth parameter, while both are small, the unperturbed state no longer is required to be the adiabatic state. The motion of a liquid in this case is governed by the Boussinesq equations. For the case of a weakly heated, shallow gas, the relation between perturbation temperature and perturbation density depends on the conditions under which the gas is maintained. Specific results are given for an ideal gas maintained at constant pressure to order ε [equation (47a)], and at constant volume to order ε [equation (47b)].

All sets of equations mentioned can account for slow 'internal' waves. None of the sets contains acoustic waves; the absence of these is a consequence of the vanishingly small ratio of the characteristic velocity β to the sound velocity. This greatly reduces the difficulty of solving the equations numerically, as compared with solving the full compressible equations [19–21]. The main reason for the difference is that numerical schemes that are explicit in time no longer are subject to the Courant–Friedrichs–Lewy condition on the time step (cf. Roache [28]). As a result, numerical solutions are no more difficult to obtain than solutions of incompressible flow equations.

The present analysis is believed to be useful in the following respects:

(1) It organizes past work on the subject in a systematic way, clarifying various differences and similarities. Several new results are obtained in the process.

(2) All sets of equations derived allow the use of simple differential schemes of numerical solution.

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APPENDIX

COMPARISON WITH PREVIOUS RESULTS

The work of Oberbeck [1] essentially concerns a liquid for which $\alpha_m \Delta\theta \ll 1$. Oberbeck neglected terms of order $\alpha_m \Delta\theta$ in the momentum and energy equations, but kept a term of that order in the continuity equation. Other simplifications in Oberbeck's treatment are the neglect of the temperature dependence of the fluid properties, and the assumption of steady-state conditions. Apart from these points, Oberbeck's equations (7)–(9) correspond to equations (54)–(58) of Section 6 with $K_m p_m$ set equal to zero. In an attempt to solve these equations, Oberbeck [1] expanded all quantities in powers of α. He assumed that the expansion of the dimensional velocity starts with a term of order α. The present relation $\beta = (\delta g L)^{1/2}$ shows that this assumption is invalid, and that the leading term actually is of order $\alpha^{1/2}$. Although Oberbeck's assumption resulted in the appropriate form of the continuity equation, the corresponding momentum and energy equation can no longer be considered correct.

In a subsequent paper, Oberbeck [2] considered the thermally driven motion of the atmosphere. He set the temperature equal to the sum of an unperturbed part depending only on the distance to the earth's center, and a perturbation part depending also on geographical position. His resulting set of equations (3) is related to the Boussinesq approximation, but is not the same.

The well-known work of Boussinesq [3] corresponds to the case $\delta' \ll 1$, $\varepsilon \ll 1$. Boussinesq based his approximation on qualitative arguments illustrated by a numerical example concerning the terrestrial atmosphere. This numerical example corresponds to $\varepsilon \ll \delta' \ll 1$.

Rayleigh [4] was concerned with finding a theoretical explanation for the occurrence of Bénard convection cells. Rayleigh adopted Boussinesq's equations as his point of departure, and studied the growth of instability modes in a shallow layer of fluid heated from below. He divided the temperature into the sum of two parts: an unperturbed temperature proportional to height z, and a remainder representing the perturbation. The pressure was split into two analogous parts. This procedure is equivalent to the procedure mentioned following equation (48). After linearizing the resulting equations, Rayleigh studied the growth of instabilities as a function of the unperturbed temperature gradient.

Jeffreys [5] considered the form of the energy equation applicable to the instability of a compressible fluid heated from below. He showed that marginal instability occurs when the unperturbed situation is the adiabatic state. Jeffreys concluded that what matters for the study of instabilities is the excess of the actual temperature gradient over the adiabatic one.

Spiegel and Veronis [7] studied the case of a weakly heated, shallow gas. They split each of the state variables into three parts: the constant space average; the variation in the absence of motion; and the fluctuation resulting from the motion. They then considered the equivalent of the z-component of the present equation (27). The first two terms on the RHS are

$$-\frac{\partial}{\partial z}\left(\frac{P_1}{\varepsilon}\right) - \rho_0\left(\frac{P_1}{P_0} - \frac{T_1}{T_0}\right),$$

where use was made of equation (30). Spiegel and Veronis argued that in the limit of small ε, $\rho_0 P_1/P_0$ can be neglected

with respect to $(\partial/\partial z)(P_1/\varepsilon)$. This is equivalent to setting $\rho_{10} = -T_{10}$ [see equation (41c)]. Their final momentum equation (25) agrees with the present equation (39), with the provisions that the perturbation pressure equals P_{11}, and that $P_{10} = 0$. Spiegel and Veronis set $T_{10} = (B - B_{ad})z + T'_{10}$ in their equivalent to the present energy equation (40). Here, the temperature gradient B in the absence of motion is determined by the steady-state heat conduction equation including a heat source term, while B_{ad} is the adiabatic temperature gradient. A corresponding adjustment is made in ρ_{10}. Their final energy equation generalizes Jeffrey's result to flows that are unsteady and that have finite rather than infinitesimal perturbations $\bar{V}_1$ and T_1. As indicated in Section 5, certain aspects of Spiegel and Veronis's work were further elucidated by Calder [13].

Chandrasekhar [30] derived the Boussinesq equations for a shallow layer of liquid using qualitative arguments.

Mihaljan [10] considered a liquid, and introduced two small parameters: $\alpha_m\Delta\theta$ and $\varepsilon_2 = \kappa_m^2/(\rho_m^2 c_{pm}^3 L^2 \Delta\theta)$. His Boussinesq equations (4.4)–(4.6) can be derived from the present equations (54)–(58) by neglecting terms of order $K_m p_m$. As noted by Mihaljan, small $K_m p_m$ corresponds to a small depth of the liquid. While Mihaljan stated the corresponding condition to be $K_m p_m \ll \alpha_m\Delta\theta$, rederiving his results using the present equations (54)–(58) only requires $K_m p_m \ll 1$. It should also be noted that equations (54)–(58) are based on the condition $\alpha_m\Delta\theta \ll 1$, without any limitation on ε_2. In Mihaljan's treatment, the need to introduce the condition $\varepsilon_2 \ll 1$ arises from the choice of characteristic velocity $V' = \kappa_m/(\rho_m c_{pm} L)$ (V' is denoted by β in the present paper). With this choice, the coefficient $\delta\varepsilon\phi v_m/(\beta L)$ of the dissipation term in the energy equation (1) becomes $\varepsilon_2 Pr_m \Delta\theta/\theta_m$. In Mihaljan's work, the factor $\Delta\theta/\theta_m$ was removed from this coefficient by nondimensionalizing the temperature with $\Delta\theta$ rather than with θ_m as in the present paper, thus leading to the condition $\varepsilon_2 Pr_m \ll 1$. The dimensionless velocity resulting from Mihaljan's formulation is not of order 1 numerically, except for the special case $\beta = (\alpha_m\Delta\theta g L)^{1/2} \approx \kappa_m/(\rho_m c_{pm} L)$. For this case, both viscous and convective effects are of importance throughout the flow, and the condition $\varepsilon_2 Pr_m \ll 1$ becomes equivalent to the present conditions $\delta = \alpha_m\Delta\theta \ll 1$, $\varepsilon\phi\ Gr_m^{-1/2} \lesssim 1$.

Malkus [11, 12] considered the case $\delta' \ll 1$, $\varepsilon \ll 1$ for both ideal gases and liquids. He used δ' and ε as expansion parameters, and took the adiabatic state as the unperturbed situation. Malkus's results correspond to those of the present Section 5 with P_{10} set equal to zero. The absence of the terms with P_{10} in Malkus's results followed from omitting the term with P_{10} in the scaling of pressure.

Dutton and Fichtl [16] derived a set of equations applicable to a weakly heated, deep fluid, using order of magnitude considerations. Their continuity equation (4.14) and momentum equation (3.8) agree with the present equations (26) and (27), respectively. They state that the vertical variations of density and temperature in the unperturbed state are arbitrary (p. 243), which is at variance with the present equations (23b) and (24). They retained both zero- and first-order terms in the energy equation, and neglected some of the first-order terms because of the presence of analogous zero-order terms. As a result, the coefficient of the term with V_{1z} in their energy equation (5.10) is different from that in the present equation (28). The present coefficient is recovered from their equation (5.9) when all first-order terms are retained. Their equation (5.10) is less detailed than the present equation (28), because the heat conduction term, the viscous dissipation term, and the heat source term all are combined in a term representing the rate of entropy increase. Taking account of this, and also of the present zero-order solution (29), Dutton and Fichtl's energy equation (5.11) for an ideal gas nevertheless agrees with the equation for an ideal gas following from the present equation (28). This agreement does not hold for the case of a liquid.

Gough [17] considered the case of a deep fluid with small relative fluctuations in density and temperature. He invoked an unperturbed state in which the temperature gradient is maintained by a zero-order heat source distribution. This is equivalent to adding a heat source term to the present equation (23a). In deriving the perturbation equations, Gough took an average over the fluctuations. His final equations (4.15)–(4.22) combine the two orders of the mean equations and the fluctuating equations. Following Ogura and Phillips [6] and others, Gough referred to these final equations as the anelastic approximation.

The results of Dutton and Fichtl [16] and Gough [17] were extended by Kovshov [18] to the case that the reference field has time-dependent variations in the direction of gravity. Kovshov showed that between the 'anelastic' approximation and the Boussinesq approximation there is an intermediate approximation, which he called the 'approximation of mesoscale convection'. The three cases are distinguished by the relative order of magnitude of the time and height scales of the perturbations.

Gebhart [8] derived the Boussinesq equations for the case of a weakly heated, shallow fluid. He presented an order of magnitude analysis of the various terms in a nondimensionalized set of equations equivalent to the present equations (1)–(3). In the present notation, his parameter R_0 (as defined on p. 324) equals $\alpha\theta\varepsilon\phi$, while $R_1 = \alpha_m\Delta\theta$. The parameter R_2 is an indication of relative height z/L, while $R_3 = \alpha_m\theta_m\varepsilon\phi\ Pr_m$ represents the ratio of the viscous dissipation term to the thermal conduction term. The parameter R_4 is of order $\alpha_m\theta_m^2\varepsilon\phi/\Delta\theta$, and is characteristic for the magnitude of the pressure term in the energy equation. As shown in Section 5, the latter term can be neglected for a liquid under normal conditions because then $\phi \ll 1$, and also for a gas maintained at constant average pressure to order δ because then $P_{10} = 0$. Gebhart's parameter R_5 is characteristic for the magnitude of the viscous dissipation term in the energy equation. For a weakly heated fluid, this term is of order $\alpha_m\theta_m\varepsilon\phi/Gr_m^{1/2}$ in the present notation. It is of importance only when $\alpha_m\theta_m\varepsilon\phi$ is of order 1, i.e. for a deep gas or a very deep liquid [cf. equation (28)].

Plate [9] considered a weakly heated, shallow ideal gas, which he treated by a method similar to that of Spiegel and Veronis [7]. Plate took the adiabatic state as the unperturbed situation. He remarked that dropping the term equivalent to P_{10} in the present equation (41a) implies an assumption on the ratio P_{10}'/T_{10}. Plate's energy equation (2.28) still contains the viscous dissipation term, in contrast to the present equation (40). In further developments [equation (3.2) *et seq.*], Plate assumed that this term can be neglected.

Cordon and Velarde [14] reconsidered the approach followed by Mihaljan [10], and pointed out various inherent difficulties of this approach. After taking account of Malkus's treatment [11, 12], they formulated a theory for the perturbations of an adiabatic hydrostatic reference field. Cordon and Velarde presented a solution for the temperature, density and pressure of this reference field for the class of fluids obeying the relations $\tilde{\kappa} = \text{const.}$, $\tilde{c}_v K/\alpha = \text{const.}$ (here, $\tilde{}$ denotes dimensional quantities other than K, α and θ). This excludes ideal gases, for which $\tilde{c}_v K/\alpha = 1/[(\gamma - 1)\tilde{p}] \neq \text{const.}$ Cordon and Velarde's solution for the reference field is a special case of the present equations (24), (21) and (4), with one significant difference. They wrote the condition of constant entropy as $d\tilde{s}/d\tilde{z} = \tilde{c}_v\, d\theta/d\tilde{z} - (a\theta/K\tilde{\rho}^2) d\tilde{\rho}/d\tilde{z}$, but did not take account of the thermodynamic identity $d\tilde{s} = \tilde{c}_p d\theta - (\alpha\theta/\tilde{\rho})\, d\tilde{\rho}$. The latter yields the present equation (23b), which determines the temperature gradient. This contradicts Cordon and Velarde's statement that the temperature gradient can be chosen arbitrarily. As discussed by Calder [13] as well as at the end of the present Section 5, an arbitrary temperature gradient *of order* ε may indeed be added to the unperturbed state of a *shallow* fluid. The corresponding reference state is subject to the present equation (43). Similarly, an arbitrary temperature gradient *of order* δ may be added to the unperturbed state of a *weakly heated* fluid. In deriving equations for perturbations of the reference field, Cordon and Velarde used different expansion parameters for density and temperature. Their energy equation (6.30) contains the material derivative $D\rho/Dt$, which is absent from Mihaljan's equivalent result 4.6. It can be

shown that each of these equations can be derived from the present equation (40). In Mihaljan's case, this is accomplished by setting dP_{10}/dt equal to zero, while in Cordon and Velarde's case dP_{10}/dt is expressed in terms of $\partial\rho_{10}/\partial t$ and $\partial T_{10}/\partial t$, using the general equation of state $T = T(P, \rho)$. The latter procedure is universally valid, while setting dP_{10}/dt equal to zero specifies the conditions under which the gas is maintained [see also the present equations (41a)–(41c)]. It can be concluded that the latter specification is the origin of the difference in the equations mentioned. Cordon and Velarde's expansion parameters η_1 and η_2 correspond to the present parameters δ' and ε, respectively. Cordon and Velarde stressed that these parameters are of the same numerical order of magnitude for typical conditions of physical interest.

Gray and Giorgini [15] treated both gases and liquids. They noted that a meaningful velocity scale must be related to the intensity of motion, and that scales related to the molecular diffusivities such as introduced by Mihaljan [10] are unrealistic. The present equation (10) agrees with the scale introduced by Gray and Giorgini. Using this scale together with other appropriate scales, they nondimensionalized the fundamental equations of fluid dynamics, and derived the Boussinesq approximate equations by requiring eleven dimensionless parameters to be small. Eight of these involve derivatives of c_p, μ, α and κ; their smallness is equivalent to neglecting variations in the latter four quantities. The remaining three parameters are, in present notation, $\alpha_m\Delta\theta$, $\varepsilon K_m p_m$ and $\alpha_m\theta_m\varepsilon\phi$. Gray and Giorgini's 'extended' Boussinesq approximation for a liquid can be recovered from the present equations (54) by taking the limit $K_m p_m \to 0$, while retaining terms of order $\alpha_m\theta_m^2\phi/\Delta\theta$ and $\alpha_m\theta_m\phi$ (i.e. of order ϕ) in the derivation of equation (57). Gray and Giorgini point out that this approximation still contains terms representing every distinct physical mechanism that was present in the original set. Their 'strict' Boussinesq approximation results from neglecting in addition the terms with ϕ. In the present work, terms of order $K_m p_m$ and ϕ are either both retained or both neglected, on the basis of numerical considerations. If the fluid is an ideal gas, Gray and Giorgini's parameters ε_1 and ε_2 correspond to the present $\delta = \Delta\theta/\theta_m$ and $\varepsilon = gL/c_m^2$, respectively. The present extension of the Boussinesq approximation is based on $\delta \ll 1$, $\varepsilon = O(1)$, and is given by equations (26)–(30). Gray and Giorgini's extension is recovered from these equations by setting $\rho_0 = 1$, $T_0 = 1$, $\rho_1 = -T_1$ and $(\alpha\theta)_1 = c_{p1} = 0$.

The case of a strongly heated, shallow fluid has previously been considered by Foster and Emery [19], who presented sample calculations of two-dimensional incompressible and compressible free convection flows. Their equations (4) correspond to the two-dimensional form of the present equations (13)–(15a) with the heat source term set equal to zero. Their energy equation does not contain $\partial T/\partial t$, which term has been replaced by terms with dP/dt and $\partial\rho/\partial t$, using the general equation of state $T = T(P, \rho)$. They used an integral form of this energy equation to obtain the rate of pressure increase dP/dt. Forester and Emery noted that the equations do not contain acoustic waves, and described how a solution can be obtained by a finite-difference scheme. This scheme involves taking the divergence of the momentum equation, and solving the resulting elliptic equation for the perturbation pressure field. They called their computational procedure the General Elliptic Method. A similar but simplified set of equations was used by Rehm and Baum [20], who omitted the heat conduction term in the energy equation and the viscous term in the momentum equation. They stressed that the simplified equations still include important features of buoyant flows such as internal waves, while 'filtering out' high frequency sound waves. They noted that as a result numerical solution of the equations does not require the extensive computer time necessary to determine the latter waves. This point was further stressed by Paolucci [21], who derived the equations used previously by Foster and Emery [19] in a systematic way, using the Mach number as a small expansion parameter. Poalucci and Chenoweth [31, 32] presented numerical solutions of these equations applicable to the pressurized discharge of a gas from a container, and to the thermally driven motion of a gas in a channel with large prescribed temperature differences between the vertical and horizontal walls.

MOUVEMENT THERMIQUEMENT INDUIT DES FLUIDES FORTEMENT CHAUFFES

Résumé—Une méthode unifiée est utilisée pour dériver deux systèmes d'équations approchée gouvernant le mouvement thermiquement induit des fluides fortement chauffés. On tient compte de la dépendance à la température des propriétés des fluides. Les limites considérées correspondent à (a) on fluide peu profond, fortement chauffé; et (b) un liquide profond fortement chauffé avec un faible coefficient de dilatation. Deux limites adimentionnelles considérées correspondent à (c) un liquide profond et faiblement chauffé. Pour le cas (a), (c) et (d), on utilise deux paramètres de développement: la profondeur adimensionnelle ε du fluide et l'intensité adimensionnelle δ' de la distribution de la source de chaleur. Le cas (a) correspond à $\varepsilon \ll 1$; les équations correspondantes gouvernent la température, la masse volumique et la pression à un ordre peu élevé. Celui-ci est uniforme mais dépendant du temps. Le cas (c) correspond à $\delta' \ll 1$. L'état non perturbé pour ce cas est pris permanent et au repos; ceci le restreint à l'état appelé adiabatique. Quand il est appliqué à un liquide sous les conditions normales, le cas (b) conduit aux équations de Boussinesq avec des propriétés de fluide dépendant de la température. Les systèmes d'équations obtenus sont tous basés sur une vitesse caractéristique qui est extrêmement petite en comparaison de la vitesse du son. Par suite les équations ne contiennent pas d'ondes acoustiques et leur résolution numérique n'est pas plus difficile que celle des équations de l'écoulement incompressible.

THERMISCH ANGETRIEBENE BEWEGUNG STARK BEHEIZTER FLUIDE

Zusammenfassung—Eine formale vereinheitlichte Methode wird verwendet, um zwei Sätze von Näherungsgleichungen zu formulieren, welche die thermisch angetriebene Bewegung stark beheizter Fluide beschreiben. Die Temperaturabhängigkeit der Stoffwerte des Fluids wird berücksichtigt. Folgende Fälle werden betrachtet: (a) Eine stark beheizte, flache Fluidschicht und (b) eine stark beheizte, dicke Flüssigkeitsschicht mit kleinem Volumenausdehnungskoeffizienten. Zwei zusätzliche Fälle werden in Betracht gezogen: (c) Eine schwach beheizte, dicke Fluidschicht und (d) eine schwach beheizte, flache Fluidschicht. Für die Fälle (a), (c) und (d) werden zwei grundlegende Parameter verwendet: Die dimensionslose Dicke ε des Fluids und die dimensionslose Stärke δ' der Wärmequellenverteilung. Der Fall (a) entspricht $\varepsilon \ll 1$. Die resultierenden Gleichungen sind bei der Bestimmung von Temperatur, Dichte und Druck von geringster Ordnung. Der Druck ist dabei gleichförmig verteilt, aber zeitabhängig. Der Fall (c) entspricht $\delta' \ll 1$. Der ungestörte Zustand für diesen Fall ergibt sich als stationär und bewegungslos; dies beschränkt ihn auf den sogenannten adiabaten Zustand. Der Fall (d) ergibt sich aus Fall (c) unter der Annahme $\varepsilon \ll 1$. Die resultierenden Gleichungen enthalten die Boussinesq-Approximation und gelten unabhängig von der relativen Größe von δ' und ε. Wen δ' und ε von derselben Größenordnung sind, so kann ein beliebiger Temperaturgradient dieser Größenordnung dem ungestörten Zustand überlagert werden. Für den Fall einer schwach beheizten, flachen Schicht eines idealen Gases, deren Volumen konstant gehalten und gleichmäßig im Innern beheizt wird, berücksichtigen die Gleichungen den Einfluß des gestörten Druckes auf die Störung der Dichte. Der Fall (b) führt auf eine Energiegleichung, die sich ganz wesentlich von der in Fall (d) unterscheidet. Bei Anwendung auf eine Flüssigkeit unter normalen Bedingungen führt der Fall (b) zu den Boussinesq-Gleichungen mit temperaturabhängigen Fluideigenschaften. Die erhaltenen Gleichungssätze beruhen alle auf einer charakteristischen Geschwindigkeit, welche verschwindend klein im Vergleich zur Schallgeschwindigkeit ist. Als Ergebnis kann festgehalten werden, daß die Gleichungen keine akustischen Wellen enthalten und die numerische Lösung nicht schwieriger ist als bei inkompressiblen Strömungsgleichungen.

ТЕПЛОВОЕ ДВИЖЕНИЕ СИЛЬНО НАГРЕТЫХ ЖИДКОСТЕЙ

Аннотация—Для вывода двух систем приближенных уравнений, описывающих тепловое движение сильно нагретых жидкостей, используется единый формальный метод. Учитывается зависимость свойств жидкости от температуры. Исследование проводится для случаев: (а) сильно нагретого тонкого слоя жидкости; и (б) сильно нагретого слоя жидкости большой высоты с малым коэффициентом объемного расширения. Дополнительно рассматриваются случаи: (в) слабо нагретого слоя жидкости большой высоты и (г) слабо нагретого тонкого слоя жидкости. В случаях (а), (в) и (г) используются два основных параметра разложения: безразмерные толщина слоя жидкости ε и интенсивность распределения источника тепла δ'. Случай (а) соответствует значениям $\varepsilon \ll 1$; полученные уравнения позволяют определить температуру, плотность и давление. Давление в области его низших значений является однородным, но зависящим от времени. Случай (в) соответствует значениям $\delta' \leq 1$. При этом невозмущенное состояние считается стационарным и устойчивым, что позволяет принимать его за так называемое адиабатическое. Случай (г) вытекает из результатов случая (в) также при $\varepsilon \ll 1$. Результирующие уравнения представляют собой приближение Буссинеска и являются справедливыми независимо от относительного значения величин δ' и ε. Если δ' и ε имеют один и тот же порядок, то в возмущенное состояние можно дополнительно ввести произвольный градиент температуры того же порядка. В случае слабо нагреваемого тонкого слоя идеального газа постоянного объема в уравнениях учитывается влияние возмущения давления на возмущение плотности и нагрев при сжатии. В случае (б) получено уравнение энергии, которое существенно отличается от уравнения для случая (г). Если использовать случай (б) для жидкости при обычных условиях, можно получить уравнение Буссинеска с зависящими от температуры свойствами жидкости. При выводе системы уравнений предполагалось, что характеристическая скорость пренебрежимо мала по сравнению со скоростью звука. Поэтому уравнения не включают акустические волны и их численное решение представляет не большую сложность, чем решение уравнений для несжимаемой жидкости.